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ECE 204 *Numerical methods*

Summary of the tools for finding algorithms and looking ahead

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Introduction

- In this topic, we will
  - Review the seven tools we will use to find algorithms for numerical algorithms
  - Look at the upcoming four topics

Weighted averages

- To begin, the first tool is weighted averages:
  - If $w_1 + w_2 + \cdots + w_n = 1$ are weights, then

$$w_1 x_1 + w_2 x_2 + \cdots + w_n x_n$$

    is a weighted average of the n x-values
  - If all the weights are greater-than-or-equal to zero, we call it a *convex combination*, and

$$\min\{x_1, x_2, \ldots, x_n\} \le w_1 x_1 + w_2 x_2 + \cdots + w_n x_n \le \max\{x_1, x_2, \ldots, x_n\}$$

Iteration

- The second tool is iteration:
  - It is possible to devise algorithms that can be implemented as a function F so that if x_0 approximates a value,
    then, under certain conditions, $F(x_0)$ is a better approximation
  - Such an algorithm can be iterated
- Specifically, if we are solving $x = f(x)$ where f is an algebraic expression in x, we can proceed as follows:
  - Start with an initial approximation x_0
  - Let $x_{k+1} \leftarrow f(x_k)$
  - If this sequence converges, it converges to a solution of $x = f(x)$
- We saw that it is necessary to limit the number of iterations:
  - If there are too many iterations, we should halt
  - If $|x_{k+1} - x_k|$ is sufficiently small, assume x_{k+1} is close enough

Linear algebra

- The third tool is linear algebra:
  - Solving a system of linear equations $A\mathbf{u} = \mathbf{v}$ is the only such system that can be generally solved
    - We will approximate non-linear equations with linear ones
  - When applying Gaussian elimination, it is necessary to use partial pivoting to avoid adding a large multiple of one row onto another
  - It is also possible to rewrite $A\mathbf{u} = \mathbf{v}$ in the form
    $$\mathbf{u} = A_{\text{diag}}^{-1}\left(\mathbf{v} - A_{\text{off}}\mathbf{u}\right)$$
    which can then be iterated
    - This will converge if A is strictly diagonally dominant
  - We discussed the condition number of a matrix
    - Any error in A, $\mathbf{v}$ or numeric error in solving is potentially magnified by up to the condition number

Interpolation

- The fourth tool is interpolation:
  - Given n points, we can always find a unique polynomial of degree $n - 1$ that passes though all n points
    - This requires the x values are all unique
  - This requires the generation of a Vandermonde matrix
  - The condition number of the Vandermonde matrix is smaller if:
    - The x-values are closer to the origin
    - The x-values are equally spaced
    - The number of points n is kept small
  - The former can be achieved by shifting the x-values towards the origin by subtracting off their average

Taylor series

- The fifth tool is Taylor series:
  - For an appropriately differentiable function, the Taylor series gives both the approximation and the error
  - We will write Taylor series in the form:

$$f(x+h) = f(x) + f^{(1)}(x)h + \frac{1}{2} f^{(2)}(\xi)h^2$$

  - An n^{th}-order Taylor series approximates f with a polynomial of degree n together with an error term multiplied by h^{n+1}
    - We will say the error is $O(h^{n+1})$
  - We will use Taylor series both to analyze the errors in our approximations, the rates of converges of various iterative algorithms, and motivate solutions other analytic problems

Bracketing

- The sixth tool is bracketing:
  - If nothing else, if we can constrain a solution to lie on an interval $[a, b]$, it may be possible to devise an algorithm to reduce the interval on which the solution exists
    - We may, therefore, be able to reduce the range of values over which a solution exists
  - We will call this *bracketing* the solution,
    but this is not common in the literature

Intermediate-value theorem

- The seventh tool is the intermediate-value theorem:
  - If f is a continues function defined on the interval $[a, b]$, then
    - The function f takes on all values between $f(a)$ and $f(b)$ for some value of x on that interval
    - The function f also takes on all values between the extreme points the function achieves on the interval $[a, b]$
  - This will be used to determine the error of many of our algorithms

Looking ahead

- We have now described the tools we will use
  - In the next four topics, we will apply these tools to solve problems associated with
    - Approximate values of algebraic and analytic expressions
    - Approximating solutions to algebraic equations and systems of algebraic equations
      - This includes both linear equations and non-linear equations
    - Approximating solutions to analytic equations and systems of analytic equations
      - This includes initial- and boundary-value problems for both ordinary and partial differential equations
    - Approximating solutions to optimization problems

Summary

- Following this topic, you now
  - Reviewed the seven tools we will use
  - Described the next four topics in the course

References

[1] https://en.wikipedia.org/wiki/Numerical_analysis

Acknowledgments

None so far.

Colophon

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